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Some Prosperous Outcomes In 2- Banach Space For Fixed And Common Fixed Point Theorems

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Abstract

In this present review article we have explored back to back theorems in 2- Banach space for unique fixed point and common fixed point with a contractive type condition which was the enhancement of well known results .

Key Words : - Banach spaces, Fixed point theorem , Common fixed point theorem, Identity mapping , Expansion mappings .

1. Introduction

The notion of 2-Banach space was initiated by Gahler [7,8] in the year 1965 and enhanced by Iseki [9,10] with obtaining some results on fixed point theorems in 2-Banach spaces. After it many scientists , researchers and mathematicians have improvised , enhanced and demonstrated many fruitful results using various type of inequalities. By referring Brouwder [3] result it carried on by Sayyed et al [20,21], Sayyed and Badshah [19] and Jain and Sayyed [11]with various type of contractive conditions and found similar results which was used in this article . Continuing the same sequence in 2- Banach space authors namely white [26], Ahmed and Shakil [1], Khan & Imdad [13], Qureshi and Singh [15], Badshah & Gupta [2], Choudhary & Malviya [4], Som [23], Jong [12] and Datson [5] . A short time ago a full groom in 2- Banach space by Yadav et al [27], Dwavedi et.al.[6], Utpalendu and Hora Krishna [24] Saluja and Dhakde [16], Saluja [17], Malceski & Anevska [14], Vijayvargiya and Bharti [25], Shrivas [22] and Sarkar et.al.[18] with more significant and fertile result for development of advance Mathematics .

2. PRELIMINARY

In this article we shall use the following definitions which was defined by Gahler [7,8] .

DEFINITION 2.1: Let X be a linear space and $||.,.||$ is a real valued function defined on X where

(i) $||a, b|| = 0$ if and only if a and b are linearly dependent ,

(ii) $|| a, b || = || b, a ||$,

(iii) $|| a, xb || = |x| \, || a, b ||$,

(iv) $|| a, b+c || \leq || a ,b || + || a,c ||$

For all a ,b, c ϵ X and x ϵ R. Then || .,. || is called a 2-norm and the pair (X , || .,. ||) is called a 2-norm space.

REMARK 2.1: In whole review article we denote X as a 2-normed space unless otherwise stated.

DEFINITION 2..2 : A sequence $\{x_n\}$ in a 2-norm space X is said to be convergent if there is a point x ϵ X such that $\lim_{n\to\infty} || x_n - x, a || = 0$ for all a ϵ X.

DEFINITION 2.3: A sequence $\{x_n\}$ in a 2-norm space X is called a Cauchy sequence if $\lim_{n,m\to\infty} || x_n - x_m, a || = 0$ for all a ϵ X.

DEFINITION 2.4: A linear 2-norm space is said to be complete if every Cauchy sequence in X is convergent in X. Then we say X is a 2-Banach Space.

3. MAIN RESULTS

THEOREM 3.1 : Let U be a mapping of a 2- Banach space and itself, if U satisfies the following conditions :

$U^2 = I$ (Identity mapping) --- (A)

$$||Ux - Uy , a|| \geq p\,[||x - Ux ,a||\, ||y - Uy , a|| + ||y - Ux , a||\, ||x -Uy , a||] \,/\, ||x - y , a||$$
$$+ q\,[||x - Ux ,a||\, ||x - Uy , a|| + ||y - Uy , a||\, ||y - Ux , a||] \,/\, ||x -y , a||$$
$$+ r\,[||x - Ux ,a|| + ||y - Uy , a||] + r'\, ||x - y ,a||$$

--- (A')

Where $x \neq y$ and p , q , r and r' are non negative with $0 \leq 5p + 4q + 4r + r' > 2$, then U has a unique fixed point .

PROOF : Suppose x is any point in 2- Banach space and taking $y = \frac{1}{2}(U + I)x$ and $z = U(y)$ with using equation (A') ,we get

$$||z - x , a|| = ||Uy - U^2x , a|| = ||Uy - U(Ux) , a ||$$
$$\geq p\,[||y - Uy ,a||\, ||Ux - U(Ux) , a|| + ||Ux - Uy , a||\, ||y -U(Ux) , a||] \,/\, ||x - y , a||$$
$$+ q\,[||y - Uy ,a||\, ||y - U(Ux) , a|| + ||Ux - U(Ux) , a||\, ||Ux - Uy , a||] \,/\, ||x -y , a||$$
$$+ r\,[||Ux - U(Ux) ,a|| + ||y - Uy , a||] + r'\, ||y - Ux ,a||$$

By using (A) and assumed condition , we write

$$\geq p\,[||y - Uy ,a||\, ||Ux - x , a|| + \tfrac{1}{4}||Ux - x , a||^2] \,/\, \tfrac{1}{2}\, ||Ux - x , a||$$
$$+ q\,[||y - Uy ,a||\, \tfrac{1}{2}||Ux - x) , a|| + ||Ux - x) , a||\, \tfrac{1}{2}||Ux - x , a||] \,/\, \tfrac{1}{2}|| Ux -x , a||$$

$+ r\ [||Ux - x ,a|| + ||y - Uy , a||] + r' \frac{1}{2} ||x - Ux ,a||$

Or

$||z - x , a|| \geq p [2||y - Uy ,a|| | + \frac{1}{2} ||Ux - x , a||]$

$+ q [||y - Uy ,a|| + ||Ux - x) , a||] |$

$+ r\ [||Ux - x ,a|| + ||y - Uy , a||] + r' \frac{1}{2} ||x - Ux ,a||$

Or

$||z - x , a|| \geq (2p + q + r) ||y - Uy ,a|| | + (\frac{1}{2} p + q +r +\frac{1}{2} r') ||Ux - x , a||$ --- (A'')

Now for ,

$||u - x ,a|| = ||2y - z - x , a|| = ||Ux - Uy , a ||$

Using equation (A') , we get

$\geq p [||x - Ux ,a|| ||y - Uy , a|| + ||y - Ux , a|| ||x -Uy , a||] / ||x - y , a||$

$+ q [||x - Ux ,a|| ||x - Uy , a|| + ||y - Uy , a|| ||y - Ux , a||] / ||x -y , a||$

$+ r\ [||x - Ux ,a|| + ||y - Uy , a||] + r' ||x - y ,a||$

$\geq p [||x - Ux ,a|| ||y - Uy , a|| + \frac{1}{2}||x - Ux , a|| \frac{1}{2}||x -Ux , a||] / \frac{1}{2} ||x - Ux , a||$

$+ q [||x - Ux ,a|| \frac{1}{2} ||x - Ux , a|| + ||y - Uy , a|| \frac{1}{2}||x - Ux , a||] / \frac{1}{2}||Ux -x , a||$

$+ r\ [||x - Ux ,a|| + ||y - Uy , a||] + r' ||x - y ,a||$

Or

$\geq p [2 ||y - Uy , a|| + \frac{1}{2}||x -Ux , a||] + q [||x - Ux ,a|| + ||y - Uy , a||]$

$+ r\ [||x - Ux ,a|| + ||y - Uy , a||] + r' \frac{1}{2}||Ux -x , a||$

Or

$||u - x , a|| \geq (2p + q + r) ||y - Uy , a|| + (\frac{1}{2}p + q + r + \frac{r'}{2}) ||x -Ux , a||$ --- (A''')

Now

$||z - u , a|| = ||z - x ,a|| + || x - u ,a||$, then by (A'') and (A''') , we have

$||z - u , a|| \geq (4p + 2q + 2r) ||y - Uy , a|| + (p + 2q + 2r + r') ||x -Ux , a||$

--- (A'''')

On other hand

$||z - u , a|| = || Uy - (2y -z) ,a||$

$= || Uy - 2y + Uy , a||$

$= 2 ||Uy - y ,a||$ --- (A^v)

So

$2 ||Uy - y ,a|| \geq (4p + 2q + 2r) ||y - Uy , a|| + (p + 2q + 2r + r') ||x -Ux , a||$

$(2 - 4p - 2q - 2r.) ||y - Uy , a|| \geq (p + 2q + 2r + r') ||x -Ux , a||$

Or

$||x - Ux ,a|| \leq \frac{2-4p-2q-2r}{p+2q+2r+r'} ||y - Uy ,a ||$

Or $||x - Ux ,a|| \leq k ||y - Uy ,a ||$, where $k = \frac{2-4p-2q-2r}{p+2q+2r+r'} < 1$.

Let $R = \frac{1}{2}(U + I)$, then

$||R^2x - Rx , a|| = || RR(x) - Rx , a|| = ||Ry - y , a|| = \frac{1}{2} ||y - Uy ,a|| < \frac{k}{2} ||x - Ux ,a||$

by the definition of R we claim that $\{R^nx\}$ is a Cauchy sequence in X . $\{R^nx\}$ is converges to a element x_0 in X , so $\lim_{n\to\infty} \{R^nx\} = x_0$, so $\{Rx_0\} = x_0$. Hence $Ux_0 = x_0$.

UNIQUENESS : If possible $y_0 \neq x_0$ is a another fixed point of U, then

$||x_0 - y_0 , a|| = ||Ux_0 - Uy_0 , a||$, then by using equation (A') , we have

$\geq p [||x_0 - Ux_0 ,a|| || y_0 - Uy_0 , a|| + || y_0 - U x_0 , a|| || x_0 -Uy_0 , a||] / || x_0 - y_0 , a||$

$+ q [||x_0 - Ux_0 ,a|| || x_0 - Uy_0 , a|| + || y_0 - Uy_0 , a|| || y_0 - U x_0 , a||] / || x_0 - y_0 , a||$

$+ r [||x_0 - Ux_0 ,a|| + || y_0 - Uy_0 , a||] + r' || x_0 - y_0 ,a||$

Or

$|| x_0 - y_0 ,a|| \geq (p + r') || x_0 - y_0 ,a||$, which is a contradiction , hence $y_0 = x_0$. It is clear that fixed point is unique.

THEOREM 3.2 : Let U and V be two expansion mappings of a 2- Banach space X into itself and U and V satisfying the following conditions ,

(C-1) U and V are commute ,

(C-2) $U^2 = I$ and $V^2 = I$, where I is identity mapping

(C-3) $|| Ux - Uy , a|| \geq p [|| Vx - Ux ,a|| \, ||Vy - Uy , a|| + ||Vy - Ux , a|| \, ||Vx -Uy , a||] / ||Vx - Vy,a||$

$+ q [||Vx - Ux ,a|| \, ||Vx - Uy , a|| + ||Vy - Uy , a|| \, ||Vy - Ux , a||] / ||Vx -Vy , a||$

$+ r [||Vx - Ux ,a|| + ||Vy - Uy , a||] + r' ||Vx -Vy ,a||$

for all $x,y \in X$, $x \neq y$ and p, q, r and r' are non negative with $0 \leq 5p + 4q + 4r + r' > 2$ and

$|| Vx - Vy || \neq 0$, then there exists a unique common fixed point of U and V such that $U(x_0) = x_0$ and $V(x_0) = x_0$.

PROOF : Suppose x is a point in 2- Banach space then clear that $(UV)^2 = I$. Now by using equation (A') , we have

$|| UV(Vx) - UV(Vy) , a|| \geq p [|| V(V^2x) - U(V^2x) ,a|| \, ||V(V^2y) - U(V^2y) , a||$

$+ ||V(V^2y) - U(V^2x) , a|| \, ||V(V^2x) -U(V^2y) , a||] / ||V(V^2x) - V(V^2y),a||$

$+ q [||V(V^2x) - U(V^2x) ,a|| \, ||V(V^2x) - U(V^2y) , a||$

$+ ||V(V^2y) - U(V^2y) , a|| \, ||V(V^2y) - U(V^2x) , a||] / ||V(V^2x) -V(V^2y) , a||$

$+ r [||V(V^2x) - U(V^2x) ,a|| + ||V(V^2y) - U(V^2y) , a||]$

$+ r' ||V(V^2x) -V(V^2y) ,a||$

$|| UV(Vx) - UV(Vy) , a|| \geq p [|| Vx - UV(Vx) ,a|| \, ||Vy - UV(Vy) , a||$

$+ ||Vy - UV(Vx) , a|| \, ||Vx -UV(Vy) , a||] / ||Vx - Vy,a||$

$+ q [||Vx - UV(Vx) ,a|| \, ||Vx - UV(Vy) , a||$

$+ ||Vy - UV(Vy) , a|| \, ||Vy - UV(Vx) , a||] / ||Vx -Vy , a||$

$+ r [||Vx - UV(Vx) ,a|| + ||Vy - UV(Vy) , a||] + r' ||Vx -Vy ,a||$

Taking Vx = e and Vy = f , then

$|| UV(e) - UV(f) , a|| \geq p [|| e - UV(e) ,a|| \, ||f - UV(f) , a||$

$+ ||f - UV(e) , a|| \, ||e -UV(f) , a||] / ||e - f, a||$

$+ q [||e - UV(e) ,a|| \, ||e - UV(f) , a||$

$+ ||f - UV(f) , a|| \, ||f - UV(e) , a||] / || e -f , a||$

$+ r [||e - UV(e) ,a|| + || f - UV(f) , a||] + r' ||e -f ,a||$

It is clear by previous theorem that R = UV has at least one fixed point say x_0 in K that is

$Rx_0 = UVx_0 = x_0$,

and $U(UV)x_0 = Ux_0$

Or $U^2 = (Vx_0) = Ux_0$

$Vx_0 = Ux_0$

now

$||Ux_0 - x_0 ,a|| = || Ux_0 - U^2x_0 , a || = || Ux_0 - U(Ux_0) ||$

$\geq p [||x_0 - Ux_0 ,a|| \, ||Ux_0 - UUx_0 , a|| + ||Ux_0 - Ux_0 , a|| \, ||x_0 -UUx_0 , a||] \, / \, ||x_0 - Ux_0 , a||$

$+ q [||x_0 - Ux_0 ,a|| \, ||x_0 - UUx_0 , a|| + ||Ux_0 - UUx_0 , a|| \, ||Ux_0 - Ux_0 , a||] \, / \, ||x_0 -Ux_0 , a||$

$+ r \, [||x_0 - Ux_0 ,a|| + ||Ux_0 - UUx_0 , a||] \, + r' \, ||x_0 - Ux_0 ,a||$

$||Ux_0 - x_0 ,a|| \geq (p + 2r + r') \, [||x_0 - Ux_0 ,a||$

Or $x_0 = Ux_0$. Hence x_0 is a fixed point in U, but $Ux_0 = Vx_0$ so, $Vx_0 = x_0$

Hence x_0 is a common fixed point of U and V.

UNIQUENESS : If possible $y_0 \neq x_0$ is a another fixed point of U and V, then

$||x_0 - y_0 , a|| = ||U^2x_0 - U^2y_0 , a|| = ||U(Ux_0) - U(Uy_0), a||$ then by using equation (A') , we have

$\geq p [|| V(Ux_0) - U(Ux_0) ,a|| \, ||V(Uy_0) - U(Uy_0) , a||$

$+ ||V(Uy_0) - U(Ux_0) , a|| \, ||V(Ux_0) -U(Uy_0) , a||] / ||V(Ux_0) - V(Uy_0),a||$

$+ q [||V(Ux_0) - U(Ux_0) ,a|| \, ||V(Ux_0) - U(Uy_0) , a||$

$+ ||V(Uy_0) - U(Uy_0) , a|| \, ||V(Uy_0) - U(Ux_0) , a||] \, / \, ||V(Ux_0) -V(Uy_0) , a||$

$+ r \, [||V(Ux_0) - U(Ux_0) ,a|| + ||V(Uy_0) - U(Uy_0) , a||]$

$+ r' \, ||V(Ux_0) -V(Uy_0) ,a||$

Or

$||x_0 - y_0 , a|| \geq (p + r') \, ||x_0 - y_0 , a||$

Hence $x_0 = y_0$, common fixed point is unique.

Or

$|| x_0 - y_0 ,a|| \geq (p + r') \, || x_0 - y_0 ,a||$, which is a contradiction , hence $y_0 = x_0$. It is clear that fixed point is unique.

CONCLUSION

In this paper, proved a unique fixed point theorem as well as common fixed point theorem by using contractive type inequality in 2-Banach space . These results can be extended to any directions and can also be extended to fixed point theory of single-valued and multivalued mappings.

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